Proofs of Restricted Shuffles

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A motivating example: Voting

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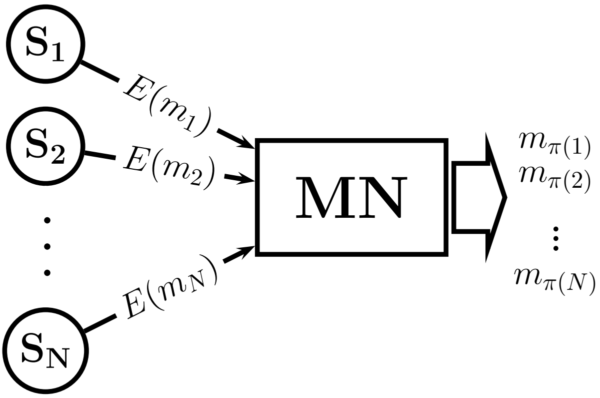
- How can we ensure that the voters remain anonymous when the votes are decrypted?
- There are two main ways to achieve this, homomorphic tallying [CGS97] and mixnets [Cha81].

Mixnets



Mixnets (2)

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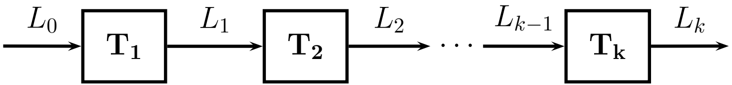
- Chain of mixservers, each permutes and re-encrypts its list of inputs.



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- Let each server produce an interactive zero-knowledge proof, a *proof of a shuffle* [SK95, Nef01, FS01].
- Like [FS01], we will construct a proof that a commitment contains a permutation matrix.
- One can then prove that the encrypted votes are permuted accordingly.

Test for permutation matrices

M permutation matrix

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

M not permutation matrix

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

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$$\prod_{i=1}^{N} \langle \overline{m}_i, \overline{x} \rangle = x_2(2x_1 - x_3)x_3 \neq x_1 x_2 x_3$$

Test for permutation matrices

Theorem (Permutation Matrix)

Let $M = (m_{i,j})$ be an $N \times N$-matrix over $\mathbb{Z}_q$ and $\overline{x} = (x_1, \ldots, x_N)$ be a list of variables. Then M is a permutation matrix if and only if

$$\prod_{i=1}^{N} \langle \overline{m}_i, \overline{x} \rangle = \prod_{i=1}^{N} x_i \quad \textit{and} \quad M\overline{1} = \overline{1} \ .$$

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Lemma (Schwartz-Zippel)

Let $f \in \mathbb{Z}_q[x_1, \ldots, x_N]$ be a non-zero polynomial of total degree d and let $e_1, \ldots, e_N$ be chosen randomly from $\mathbb{Z}_q$. Then

$$\Pr[f(e_1, \ldots, e_N) = 0] \leq \frac{d}{q} \ .$$

Recall Pedersen commitments

Let g, g_1 be randomly chosen generators in a group of prime order q. The Pedersen commitment of $m \in \mathbb{Z}_q$ is

$$\mathcal{C}(m, s) = g^s g_1^m$$

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- computationally binding
- homomorphic, $\mathcal{C}(m, s)\,\mathcal{C}(m', s') = \mathcal{C}(m + m', s + s')$
  $\mathcal{C}(m, s)^e = \mathcal{C}(em, es)$

Generalized Pedersen commitments [FS01]

Let $g, g_1, \ldots, g_N$ be randomly chosen generators in a group of prime order q. We commit to a vector $\overline{m} = (m_1, \ldots, m_N)^{\mathrm{T}}$ by

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  $\mathcal{C}\left(\overline{m}, s\right)^e = \mathcal{C}\left(e\overline{m}, es\right)$

Generalized Pedersen commitments

We commit column-wise to an $N \times N$-matrix $M = (m_{i,j})$, so $a = \mathcal{C}(M, \bar{s})$ is a list of N commitments satisfying

$$\mathcal{C}(M, \bar{s})^{\bar{e}} = \mathcal{C}(M\bar{e}, \langle \bar{s}, \bar{e} \rangle)$$

where we use the convention

$$a^{\bar{e}} = \prod_{i=1}^{N} a_i^{e_i} \quad .$$

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1. the view of the verifier can be simulated for any given challenge
2. a witness can be computed from any pair of accepting transcripts with the same random tape and distinct challenges

Example: Proof of knowledge of discrete logarithm

$\mathcal{P}$ wants to prove knowledge of x such that $y = g^x$

1. $\mathcal{P}$ chooses r at random and sends $\alpha = g^r$
2. $\mathcal{V}$ sends a random challenge c
3. $\mathcal{P}$ responds with $d = cx + r$

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There are similar protocols for proving any polynomial relation!

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Given a matrix commitment a, $\mathcal{P}$ wants to prove knowledge of a **permutation matrix** M and randomness $\overline{s}$ such that $a = \mathcal{C}(M, \overline{s})$.

1. $\mathcal{V}$ chooses a vector $\overline{e}$ randomly and sends it to $\mathcal{P}$.
2. $\mathcal{P}$ uses a sigma proof to prove knowledge of t, k and a vector $\overline{e}'$ such that

$$\mathcal{C}(\overline{e}', k) = a^{\overline{e}}$$
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$$\overline{e}' = M\overline{e}$$
$$\overline{1} = M\overline{1}$$
$$\prod_{i=1}^{N} \langle \overline{m}_i, \overline{e} \rangle = \prod_{i=1}^{N} e_i$$

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- The zero-knowledge property is easy.
- We must construct an extractor which computes a permutation matrix from accepting transcripts.

Sketch of proof

1. Run the extractor of the sigma proof N times with $\overline{e}_1, \ldots, \overline{e}_N$, each time extracting $\overline{e}'_i$ and k_i such that $\mathcal{C}\left(\overline{e}'_i, k_i\right) = a^{\overline{e}_i}$.

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4. Then $\sum_{j=1}^{N} \alpha_{\ell,j}\overline{e}'_j$ is the ℓth column in M since

$$a_l = \prod_{j=1}^{N} a^{\alpha_{\ell,j}\overline{e}_j} = \prod_{j=1}^{N} \mathcal{C}\left(\overline{e}'_j, k_j\right)^{\alpha_{\ell,j}} = \mathcal{C}\left(\sum_{j=1}^{N} \alpha_{\ell,j}\overline{e}'_j, \sum_{j=1}^{N} \alpha_{\ell,j}k_j\right)$$

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$$\mathcal{C}\left(\overline{e}', k\right) = a^{\overline{e}} = \mathcal{C}\left(M\overline{e}, \langle \overline{s}, \overline{e} \rangle\right)$$

but $\overline{e}' \neq M\overline{e}$.

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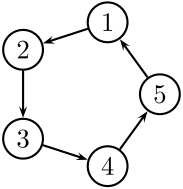
A rotation is precisely an automorphism of the directed cycle graph!



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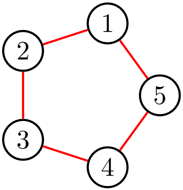

Let us look at the undirected cycle instead.

Restricting the permutation (graphs)

- Let $\mathscr{G}$ be a graph with vertices $V = \{1, 2, 3, \ldots, N\}$. Encode the edge set as

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- A permutation π is an automorphism of $\mathscr{G}$ if and only if

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- Apply Schwartz-Zippel ...

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Testing $F_{\mathscr{G}}(x_1, \ldots, x_N) = F_{\mathscr{G}}(x_{\pi(1)}, \ldots, x_{\pi(N)})$ determines whether π is a rotation.

Restricting the permutation (polynomials)

Theorem

Let F be any polynomial in $\mathbb{Z}_q[x_1, \ldots, x_N]$ and let S_F be the group of permutations π such that

$$F(x_1, \ldots, x_N) = F(x_{\pi(1)}, \ldots, x_{\pi(N)}) \enspace .$$

Then we can prove that the permutation is chosen from S_F.

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Problem Are there applications for other restrictions than rotations, e.g. automorphisms of a complete binary tree?

Questions?

References I

R. Cramer, R. Gennaro, and B. Schoenmakers.
A secure and optimally efficient multi-authority election scheme.
In *Advances in Cryptology – Eurocrypt '97*, volume 1233 of *Lecture Notes in Computer Science*, pages 103–118. Springer Verlag, 1997.

D. Chaum.
Untraceable electronic mail, return addresses and digital pseudo-nyms.
Communications of the ACM, 24(2):84–88, 1981.

References II

S. de Hoogh, B. Schoenmakers, B. Skoric, and J. Villegas.
Verifiable rotation of homomorphic encryptions.
In *Public Key Cryptography – PKC 2009*, volume 5443 of *Lecture Notes in Computer Science*, pages 393–410. Springer Verlag, 2009.

J. Furukawa and K. Sako.
An efficient scheme for proving a shuffle.
In *Advances in Cryptology – Crypto 2001*, volume 2139 of *Lecture Notes in Computer Science*, pages 368–387. Springer Verlag, 2001.

References III

A. Neff.
A verifiable secret shuffle and its application to e-voting.
In *8th ACM Conference on Computer and Communications Security (CCS)*, pages 116–125. ACM Press, 2001.

M. K. Reiter and X. Wang.
Fragile mixing.
In *11th ACM Conference on Computer and Communications Security (CCS)*, pages 227–235. ACM Press, 2004.

K. Sako and J. Killian.
Reciept-free mix-type voting scheme.
In *Advances in Cryptology – Eurocrypt '95*, volume 921 of *Lecture Notes in Computer Science*, pages 393–403. Springer Verlag, 1995.

References IV